A GENERALIZATION OF THE GAUSS-BONNET THEOREM

BY

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1. Introduction. In this paper we are concerned with relations between the characteristic classes and the curvature of a Riemann manifold X. Our main result (Theorem 4D) establishes a representation formula for the Stiefel-Whitney classes of X in terms of integrals of certain generalized Gauss and geodesic curvature forms on X.

(A) The Stiefel-Whitney classes $w^r(X)$ are cohomology classes of X of order 2 for $1 \leqq r < n = \dim X$. That these classes admit an integral representation formula is assured by the following existence theorem of Allendoerfer-Eells [3], which we state here for future reference.

A $(\mathbf{Z}, r)$-*pair* ($\mathbf{Z}$ denoting the ring of integers) *of forms* (θ, ω) *on* X consists of (1) a smooth (i.e., C^∞) $(r-1)$-form ω defined on X except perhaps for a smooth polyhedron $e(\omega)$ of dimension $\leqq n-r$, (2) a smooth r-form θ defined on X except for a subpolyhedron $e(\theta)$ of $e(\omega)$ of dimension $\leqq n-r-1$; we require that (3) $d\omega = \theta$ in $X - e(\omega)$, and (4) for any smooth *admissible* integral chain c (i.e., such that $|c| \cap e(\theta) = \varnothing$, $|\partial c| \cap e(\omega) = \varnothing$, where $|a|$ denotes the support of the chain a), the *residue* relative to c

$$(1) \qquad\qquad h(\theta, \omega) \cdot c = \int_c \theta - \int_{\partial c} \omega$$

is an integer. We say that two $(\mathbf{Z}, r)$-pairs (θ, ω) and (θ', ω') are *equivalent* if $h(\theta, \omega) \cdot c = h(\theta', \omega') \cdot c$ whenever both members are defined. The equivalence classes of $(\mathbf{Z}, r)$-pairs constitute a differential graded module $\mathfrak{C}^r(X, \mathbf{Z})$ with exterior differential defined by $d(\theta, \omega) = (0, \theta)$. The derived cohomology module $\mathfrak{H}^*(X, \mathbf{Z})$ has anti-commutative algebra structure, and equation (1) *determines a canonical isomorphism of* $\mathfrak{H}^*(X, \mathbf{Z})$ *onto the singular cohomology algebra* $H^*(X, \mathbf{Z})$ *of* X. The same theorem is valid with $\mathbf{Z}$ replaced by $\mathbf{Z}_2$ (the integers modulo 2), if we require that equivalent $(\mathbf{Z}, r)$-pairs have the same residues mod 2.

(B) Now suppose that X is orientable and has a Riemann structure. For every r ($1 \leqq r \leqq n$) we define the rth *Gauss curvature form* $\Omega^{(r)}$ and the rth *geodesic curvature form* $\Phi^{(r)}$ of X. These are smooth forms on the bundle $\mathcal{S}_{n-r}(X)$ of orthonormal $(n-r)$-frames on X (we define $\mathcal{S}^0(X) = X$); these forms are intrinsically associated with the Riemann structure of X. Furthermore, if V is an r-dimensional submanifold, then $\Omega^{(r)}$ and $\Phi^{(r-1)}$ determine the Gauss and geodesic curvature forms of V (relative to its induced Rie-

mann structure). Actually, $\Phi^{(r-1)}$ has opposite sign to the geodesic curvature form of Chern [4]; our choice is dictated by the desire to fit equation (2) below into our general residue theory.

Take any smooth $(n-r+1)$-frame $f_{r-1}(x) = (x; e_1 \cdots e_{n-r+1})$ defined on X except perhaps for a smooth $(n-r)$-polyhedron $e(f_{r-1})$. Let f_r be a smooth extension of $(x; e_1 \cdots e_{n-r})$, defined on X except for a smooth $(n-r-1)$-subpolyhedron of $e(f_{r-1})$. We will show that $(f_r^* \Omega^{(r)}, f_{r-1}^* \Phi^{(r-1)})$ *is a* $(\mathbf{Z}, r)$-*pair on* X, *and*

$$(2) \qquad w^r(X) \cdot c = \int_c f_r^* \Omega^{(r)} - \int_{\partial c} f_{r-1}^* \Phi^{(r-1)}$$

for any integral r-chain admissible for the pair. Here and henceforth (2) is to be considered a congruence modulo 2 if r is even and $<n$. In case $r=n$ equation (2) contains the Gauss-Bonnet Theorem for closed orientable Riemann manifolds (see Allendoerfer-Weil [1] and Chern [4]), because of the relation $w^n(X) \cdot X = \chi(X)$, the Euler characteristic of X. It is known (Theorem of Thom) that the Stiefel-Whitney classes are topological invariants of X; formula (2) relates them to invariants of its Riemann structure.

These integral formulas for $w^r(X)$, in a different form and derived by different methods, have been discovered by Allendoerfer [2], using forms on the $(n-r)$-skeleton of a particular cellular subdivision of X (and expressed in polar coordinates defined in each cell of that skeleton). Our formulas (2) should be considered as reinterpreting Allendoerfer's, emphasizing the geometric role played by the forms $\Omega^{(r)}$ and $\Phi^{(r-1)}$.

Also, the referee has called attention to a paper of Chern, *Integral formulas for the characteristic classes of sphere bundles*, Proc. Nat. Acad. Sci. vol. 30 (1944) pp. 269–273. In that work a closed transgressive form (apparently different (for $r<n$) from our $\Phi^{(r-1)}$) is constructed on $\mathcal{S}_{n-r+1}(X)$ for r odd, and its relation to the rth Stiefel-Whitney class of X is studied by integration over bounded $(r-1)$-submanifolds of X.

Formula (2) can be derived in two distinct ways: (1) by considering X as an abstract manifold and studying the intrinsic geometry of its frame bundles, and (2) by imbedding X in Euclidean space (of sufficiently high dimension) and considering the geometry of X induced from the classifying space—the Grassmann manifold $G_{n+m,n}$ of oriented n-planes in Euclidean $(n+m)$-space. Considering that second point of view we obtain (Theorem 3C) a set of partially invariant $(\mathbf{Z}, r)$-pairs of forms on $G_{n+m,n}$ forming a base for $H^*(G_{n+m,n}, \mathbf{Z}_2)$. I hope to present elsewhere a detailed study of invariant $(\mathbf{Z}, r)$-pairs of forms on other homogeneous spaces, having orbital singularities.

Added in proof. It has been brought to my attention that the formula (2) and the geometric interpretation of the forms $(\Omega^{(r)}, \Phi^{(r-1)})$ have been discovered by S. Takizawa, Mem. Coll. Sci. Univ. Kyoto vol. 28 (1953) pp. 1–10

and pp. 241–251. See also A. Aragnol, C.R. Acad. Sci. Paris vol. 238 (1954) pp. 2387–2389. Both authors use the method of Allendoerfer.

2. **Universal curvature forms.** (A) Let E_{n+m} denote Euclidean $(n+m)$-dimensional space with a fixed orthonormal base $e_1, \cdots, e_{n+m}$; write the coordinates of a point $x \in E_{n+m}$ as $x = (x_1, \cdots, x_{n+m})$. E_{n+m}^r is the subspace spanned by the last r vectors. Let R_{n+m} denote the rotation group in E_{n+m} and R_{n+m}^r the subgroup acting in E_{n+m}^r keeping $x_1, \cdots, x_{n+m-r}$ fixed.

For each r $(1 \leqq r \leqq n)$ let $V_{n+m,n}^r$ denote the pairs (X, f), where X is an oriented n-plane through the origin 0 in E_{n+m} and f is an $(n-r)$-frame at 0 in X; we agree that $V_{n+m,n}^n$ is the Grassmann manifold $G_{n+m,n}$ of oriented n-planes through the origin in E_{n+m}. Each $V_{n+m,n}^r$ is expressible as the factor space $R_{n+m}/R_r \times R^m$, and $V_{n+m,n}^1$ is naturally identified with the Stiefel manifold $V_{n+m,n}$ of n-frames at 0 in E_{n+m}. The coset maps define a sequence of R_n-bundles

$$V_{n+m,n} = V_{n+m,n}^1 \to \cdots \to V_{n+m,n}^r \to \cdots \to V_{n+m,n}^n = G_{n+m,n}.$$

Each composition map $V_{n+m,n}^r \to G_{n+m,n}$ *is an associated R_n-bundle of the principal bundle $V_{n+m,n} \to G_{n+m,n}$, with fibre $V_{n,n-r} = R_n/R_r$.*

(B) Maurer-Cartan forms for R_{n+m} can be taken as the entries ω_{ij} $(1 \leqq i, j \leqq n+m)$ in the skew-symmetric matrix $w = u^{-1}du$ for any $u \in R_{n+m}$, using the base in E_{n+m}. Let $L^*(R_{n+m}, R_r \times R^m)$ denote the (real) linear space of left-invariant 1-forms on R_{n+m} which are zero on the Lie algebra $L(R_r \times R^m)$; we observe that *for any r $(1 \leqq r \leqq n)$ the space $L^*(R_{n+m}, R_r \times R^m)$ is spanned by the ω_{ij} such that $(1 \leqq i \leqq n, r+1 \leqq j \leqq n+m)$.*

We will say that a left invariant form α (of any degree) on R_{n+m} is $V_{n+m,n}^r$-*basic* $(1 \leqq r \leqq n)$ if there is a form β on $V_{n+m,n}^r$ such that $\pi^*\beta = \alpha$, where π is the coset map $R_{n+m} \to V_{n+m,n}^r$; π^* is a monomorphism because the fibering is locally trivial. β is then invariant under the action of R_{n+m} on $V_{n+m,n}^r$. It is well known that the following is *a pair of necessary and sufficient conditions that α be $V_{n+m,n}^r$-basic*: (1) α *is in the exterior subalgebra generated by* $L^*(R_{n+m}, R_r \times R^m)$, *and* (2) α *is invariant under the adjoint action of* ad$(R_r \times R^m)$. If α is $V_{n+m,n}^r$-basic then we will treat α as though it were in fact defined on $V_{n+m,n}^r$.

For any $v \in R_{n+m}$ we have $(\text{ad } v)w = v^{-1}wv$; the following computations are elementary:

(1) If $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, then

$$(\text{ad } v)\omega_{ij} = \omega_{ij} \qquad \text{if } v \in R^m;$$

$$(\text{ad } v)\omega_{ij} = \sum_{p,q=1}^{n} v_{pi}\omega_{pq}v_{qj} \qquad \text{f } v \in R_n.$$

(2) If $1 \leqq i \leqq n$, $n+1 \leqq j \leqq n+m$, then

$$(ad\ v)\omega_{ij} = \sum_{k=n+1}^{n+m} \omega_{ik}v_{kj} \qquad \text{if } v \in R^m,$$

$$(ad\ v)\omega_{ij} = \sum_{k=1}^{n} v_{ki}\omega_{kj} \qquad \text{if } v \in R_n.$$

(C) DEFINITION. The $(r,\ n)$-*universal curvature matrix* is the matrix of 2-forms on R_{n+m} whose entries are the exterior polynomials

(3) $$\Omega_{ij}^{(r)} = \sum_{k=r+1}^{n+m} \omega_{ik} \vee \omega_{kj}, \qquad\qquad (1 \le i, j \le n)$$

where $\vee$ denotes exterior multiplication. The $(r,\ n)$-*universal Gauss curvature form*

(4) $$\Omega^{(r)} = \frac{(-1)^{r/2}}{2^r \pi^{r/2}(r/2)!} \sum \epsilon_{i_1 \ldots i_r}\Omega_{i_1 i_2}^{(r)} \vee \cdots \vee \Omega_{i_{r-1} i_r}^{(r)} \qquad \text{if } r \text{ is even,}$$

$$= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } r \text{ is odd;}$$

the summation is taken over all selections $(i_1 \cdots i_r)$ from $(1 \cdots n)$. We will see in §4 that the $\Omega_{ij}^{(r)}$ determine the curvature matrix for every r-dimensional submanifold of a Riemann n-dimensional submanifold of E_{n+m}.

It is well known (and easily seen) that the forms ω_{ij} ($1 \le i,\ j \le n$) are $V_{n+m,n}$-basic and are the forms of an affine connection for the principal R_n-bundle $V_{n+m,n} \to G_{n+m,n}$; its curvature matrix has the entires (3) with $r = n$. Furthermore, $\Omega^{(n)}$ is $G_{n+m,n}$-basic and is the *universal Gauss-Bonnet form*; see Chern [6].

Consider now the corresponding situation for the associated bundles.

LEMMA. $\Omega^{(r)}$ *is* $V_{n+m,n}^r$-*basic; thus it can be considered as an* R_{n+m}-*invariant form on* $V_{n+m,n}^r$.

Proof. Clearly $\Omega^{(r)}$ is in the subalgebra generated by $L^*(R_{n+m},\ R_r \times R^m)$. Applying (1) and (2) we see that $(ad\ v)\Omega_{ij}^{(r)} = \Omega_{ij}^{(r)}$ for any $v \in R^m$, whence $(ad\ v)\Omega^{(r)} = \Omega^{(r)}$. Similarly, if $v \in R_r$ then

$$(ad\ v)\Omega_{ij}^{(r)} = \sum_{p,q=1}^{n} v_{pi}v_{qj}\Omega_{pq}^{(r)};$$

the determinant identity $\epsilon_{i_1 \ldots i_r}v_{p_1 i_1} \cdots v_{p_r i_r} = \epsilon_{p_1 \ldots p_r}$ for $v \in R_r$ shows that $(ad\ v)\Omega^{(r)} = \Omega^{(r)}$. The lemma now follows from the result quoted in (B).

(D) A key step in Chern's proof [4] of the Gauss-Bonnet Theorem shows that the curvature form Ω of an orientable Riemann manifold X is derived in the tangent sphere bundle $\pi: \mathcal{S}(X) \to X$; i.e., there is an $(r-1)$-form (the geodesic curvature form) Φ on $\mathcal{S}(X)$ such that $\pi^*\Omega = d\Phi$. (As remarked in our Introduction, Chern actually uses $-\Phi$.) Furthermore, the restriction of Φ to a fibre S_{n-1} of $\mathcal{S}(X)$ determines (in the sense of de Rham's Theorem) a gen-

erator for $H^{n-1}(S_{n-1}, \mathbf{Z})$. These properties are sometimes expressed by saying that (Ω, Φ) *is a transgressive pair of forms in the sphere bundle of* X. Chern's method also gives a universally transgressive pair $(\Omega^{(n)}, \Phi^{(n-1)})$ in the sphere bundle $V_{n+m,n}^{n-1} \to G_{n+m,n}$. We now show that a similar result (with the same type of construction) holds for every r $(2 \leqq r \leqq n)$.

If $[r/2]$ denotes the largest integer $\leqq r/2$, then for all k $(0 \leqq k \leqq [r/2]-1)$ we define the exterior polynomials on R_{n+m}

$$\Phi_k^{(r-1)} = \sum \epsilon_{i_1 \cdots i_{r-1}} \Omega_{i_1 i_2}^{(r)} \vee \cdots \vee \Omega_{i_{2k-1} i_{2k}}^{(r)} \vee \omega_{i_{2k+1} r} \vee \cdots \vee \omega_{i_{r-1} r},$$

$$\Psi_k^{(r-1)} = 2(k+1) \sum \epsilon_{i_1 \cdots i_{r-1}} \Omega_{i_1 i_2}^{(r)} \vee \cdots \vee \Omega_{i_{2k+1} r}^{(r)} \vee \omega_{i_{2k+2} r} \vee \cdots \vee \omega_{i_{r-1} r}.$$

Set $\Psi_{-1}^{(r-1)} = \Psi_{[r/2]}^{(r-1)} = 0$, and observe that $\Phi_{[r-2]}^{(r-1)}$ is defined if r is odd. (Superscripts do not necessarily denote degrees!)

LEMMA. *If* $2 \leqq r \leqq n$ *and* $0 \leqq k \leqq [r/2]-1$, *then the forms* $\Phi_k^{(r-1)}$ *and* $\Psi_k^{(r-1)}$ *are* $V_{n+m,n}^{r-1}$-*basic and satisfy*

$$(5) \qquad d\Phi_k^{(r-1)} = \Psi_{k-1}^{(r-1)} + \frac{r-2k-1}{2(k+1)} \Psi_k^{(r-1)}.$$

The proof that the forms are basic is just like that of Lemma 2C. The proof of (5)—computed in $V_{n+m,n}$—follows Chern's proof [4] (see also Flanders [7] for a detailed exposition using vector valued forms) for the case $r=n$. Set

$$(6) \qquad a_k = \frac{(-1)^{r+k-1}}{2^r \pi^{(r-1)/2} k! \, \Gamma\left(\dfrac{r-2k+1}{2}\right)}$$

and

$$(7) \qquad \Phi^{(r-1)} = \sum_{k=0}^{[(r-1)/2]} a_k \Phi_k^{(r-1)} \qquad \text{for } 2 \leq r \leq n.$$

Then $\Phi^{(r-1)}$ is an R_{n+m}-invariant form on $V_{n+m,n}^{r-1}$, which we call the $(r-1, n)$-*universal geodesic curvature form*; by combining (5) and (7) we obtain

$$(8) \qquad d\Phi^{(r-1)} = \Omega^{(r)}.$$

In particular, $\Phi^{(r-1)}$ is closed if r is odd.

The bundle $V_{n+m,n}^{r-1} \to V_{n+m,n}^r$ has typical fibre $S_{r-1} = R_r/R_{r-1}$. The restriction of $\Phi^{(r-1)}$ to S_{r-1} is $a_0 \Phi_0^{(r-1)}$, which determines a generator of $H^{r-1}(S_{r-1}, \mathbf{Z})$. We summarize these results in the

PROPOSITION. *Considered as* R_{n+m}-*invariant forms on* $V_{n+m,n}^r$ *and* $V_{n+m,n}^{r-1}$ *respectively,* $(\Omega^{(r)}, \Phi^{(r-1)})$ *is a transgressive pair in the* $(r-1)$-*sphere bundle* $V_{n+m,n}^{r-1} \to V_{n+m,n}^r$.

3. R_n-**universal Stiefel-Whitney classes.** (A) We will suppose henceforth that $m \geq n+1$. Any sequence of linear subspaces $0 \subset L_1 \subset \cdots \subset L_{n+m} = E_{n+m}$ (with subscripts denoting dimensions; for exposition see Chern [5] or Wu [10]) determines an oriented cellular subdivision of the Grassmann manifold $G_{n+m,n}$, defined in terms of the Schubert varieties $(a_1 \cdots a_n) = \{X \in G_{n+m,n}: \dim (X \cap L_{a_i+i}) \geq i, \ (i \leq i \leq n)\}$, where the a_i are integers satisfying $0 \leq a_1 \leq \cdots \leq a_n \leq m$. Each $(a_1 \cdots a_n)$ is a circuit (pseudomanifold) of dimension $a_1 + \cdots + a_n$, expressible as the closure of the union of two open oriented cells $(a_1 \cdots a_n)^+$ and $(a_1 \cdots a_n)^-$. We will let $(a_1 \cdots a_n)^\pm$ also stand for the elementary cochain (with integer or integer mod 2 coefficients) which is $+1$ on the chain $(a_1 \cdots a_n)^\pm$ and zero on other cells.

For each r $(1 \leq r \leq n)$ let w_r denote the chain

$$w_r = (\underset{\leftarrow \ r \ \rightarrow}{0 \cdots 0 \ 1 \cdots 1})^+ - (\underset{\leftarrow \ r \ \rightarrow}{0 \cdots 0 \ 1 \cdots 1})^-,$$

and let w^r denote the corresponding cochain. It follows from general boundary-coboundary formulas for Schubert varieties that

(1) $$\partial w_r = [1 + (-1)^{r+1}]w_{r-1}, \qquad dw^r = [1 + (-1)^r]w^{r+1}.$$

In particular, w^r is a cocycle if r is odd or $r = n$, and w^r is a cocycle mod 2 if r is even.

The R_n-*universal Stiefel-Whitney classes* are the cohomology classes $(w^r) \in H^r(G_{n+m,n}, \mathbf{Z})$ of w^r if r is odd or $r = n$, and $(w^r) \in H^r(G_{n+m,n}, \mathbf{Z}_2)$ if r is even and $r < n$. These classes are independent of the particular sequence of linear subspaces chosen to define the subdivision and of $m \geq n+1$. The class (w^n) is of infinite order and is sometimes called the *Euler-Poincaré class.* The other Stiefel-Whitney classes have order 2.

REMARK. Let x^{r-1} denote a generator of $H^{r-1}(V_{n,n-r+1}, \mathbf{Z})$ if r is odd or $r = n$, and of $H^{r-1}(V_{n,n-r+1}, \mathbf{Z}_2)$ if r is even and $r < n$. Since $V_{n,n-r+1}$ is $(r-2)$-connected it follows that x^{r-1} *is transgressive in the bundle* $V_{n+m,n}^{r-1} \to G_{n+m,n}$. Using the technique in (C) below Pontrjagin has shown that *the unique transgressive image of* x^{r-1} *is the rth universal Stiefel-Whitney class.*

(B) Let us consider the oriented subdivision K of $G_{n+m,n}$ defined by the linear spaces

$$L_1 = \{x = (0, \cdots, 0, x_{n+1}, 0, \cdots, 0)\},$$

$$L_2 = \{x = (0, \cdots, 0, x_n, x_{n+1}, 0 \cdots 0)\}, \ \cdots,$$

$$L_{n+2} = \{x = (x_1, \cdots, x_{n+2}, 0, \cdots, 0)\}, \ \cdots, L_{n+m} = E_{n+m}.$$

If in the definition of Schubert varieties we replace L_{a_i+i} by L^{a_i+i} (letting L^p denote the oriented orthogonal complement of L_{n+m-p}), we obtain an oriented dual subdivision K_* of $G_{n+m,n}$. Using stars to indicate the cells of K_*, the dual $(nm-r)$-cycle (or cycle mod 2) $\mathfrak{D}w^r$ of w^r is given by

$$\mathfrak{D}w^r = (-1)^\lambda [(\underleftarrow{m-1}\cdots\overrightarrow{m-1}\,m\cdots m)_*^+$$

(2)
$$- (-1)^{n+m}(m-1\cdots m-1\,m\cdots m)_*^-]$$

if $n(n-1)/2$ is even, and

$$= (-1)^{\lambda+1}[(-1)^{n+m}(m-1\cdots m-1\,m\cdots m)_*^+$$

$$- (m-1\cdots m-1\,m\cdots m)_*^-]$$

if $n(n-1)/2$ is odd, where $(-1)^\lambda$ is the algebraic number of intersections of the dual cells $(0\cdots0\,1\cdots1)^+$ (r ones) and $(m-1\cdots m-1\,m\cdots m)_*^+$; see Wu [10, Chapter I, p.3].

(C) DEFINITION. Relative to the dual subdivisions K and K_* of $G_{n+m,n}$ described in (B) we define an *admissible pair of sections*

$$f_p;\ G_{n+m,n} - (m-1\cdots m-1\,m\cdots m)_* \to V_{n+m,n}^p$$

$$\underleftarrow{\quad p+1 \quad}\rightarrow$$

as a pair of smooth sections ($p=r-1,\,r$) such that $\pi\circ f_{r-1}=f_r$, where π is the bundle map $\pi\colon V_{n+m,n}^{r-1}\to V_{n+m,n}^r$. Admissible pairs can be constructed for all r following a method of Pontrjagin [8, §1D]; see also Chern [5, p. 93]. We will suppose that $2\leqq r\leqq n$; the case $r=1$ is trivial.

THEOREM. *Let $\Omega^{(r)}$ and $\Phi^{(r)}$ denote the $(r,\,n)$-universal Gauss and geodesic curvature forms; set*

$$\omega = f_r^*\Omega^{(r)}\quad and\quad \phi = f_{r-1}^*\Phi^{(r-1)}.$$

Then $(\omega,\,\phi)$ is a $(\mathbf{Z},\,r)$-pair $(2\leqq r\leqq n)$ on $G_{n+m,n}$ whose singularities are the supports of $\partial\mathfrak{D}w^r$ and $\mathfrak{D}w^r$ respectively. For any integral r-chain a_r of K we have

(3)
$$w^r\cdot a_r = \int_{a_r}\omega - \int_{\partial a_r}\phi.$$

In particular, the residues of $(\omega,\,\phi)$ are independent of the choice of admissible pair of sections used to define them.

Proof. The statement about the singularities follows immediately from (B). In its domain of definition we have $df_{r-1}^*\Phi^{(r-1)}=f_{r-1}^*\circ\pi^*\Omega^{(r)}=f_r^*\Omega^{(r)}$, and therefore $d\phi=\omega$. To show that the pair has integral residues (on all admissible integral chains) it suffices to verify (3) for the integral r-chains of K, by Allendoerfer-Eells [3, Corollary 5A].

We see that $w^r\cdot a_r$ is zero on the cells of K except when a_r is $\sigma_r(\pm)$ $=(0\cdots0\,1\cdots1)^\pm$ (r ones), in which case $w^r\cdot\sigma_r(\pm)=\pm1$. On the other hand, if $a_r\neq\sigma_r(\pm)$ then a_r and $\mathfrak{D}w^r$ do not intersect, whence we can apply Stokes' Theorem to conclude that the right member of (3) is zero; otherwise we find that

$$(4) \qquad \int_{\sigma_r(\pm)} \omega - \int_{\partial\sigma_r(\pm)} \phi = \int_{\sigma_r(\pm)} \omega + \int_{\sigma_{r-1}(\pm)} \phi + \int_{\sigma_{r-1}(\mp)} \phi \qquad \text{if } r \text{ is even, and}$$

$$= - \int_{\sigma_{r-1}(\pm)} \phi + \int_{\sigma_{r-1}(\mp)} \phi \qquad \text{if } r \text{ is odd.}$$

Now the Schubert variety $(0 \cdots 0\ 1 \cdots 1) = \{X \in G_{n+m,n}: L_{n-r} \subset X \subset L_{n+1}\}$ is naturally identified with the unit r-sphere in E_{r+1}, and $\sigma_r(\pm)$ are complementary oriented hemispheres. By considering the restrictions of (4) and (7) of §2 to the parts of the associated bundles $V_{n+m,n}^p$ over the integration domains of (4), we see that on those domains ω and ϕ are, independently of the choice of the admissible pair of sections,

$$\frac{\Gamma\left(\dfrac{r+1}{2}\right)}{\pi^{(r+1)/2}} \, \omega_{1,r+1} \vee \cdots \vee \omega_{r,r+1}$$

and

$$\frac{\Gamma\left(\dfrac{r}{2}\right)}{2\pi^{r/2}} \, \omega_{1,r} \vee \cdots \vee \omega_{r-1,r}.$$

Considering the orientation conventions (Wu [10, p. 17]) for the cells of K, we find that the right members of (4) equal one. Thus (3) is established in every case, and the proof is complete. Compare this proof with that of Chern's representation formula [5, p. 77] for the universal Chern classes.

By the isomorphism theorem quoted in §1A we obtain the

COROLLARY. *The cohomology class with* **Z** *(resp.* **Z₂**) *coefficients if r is odd or $r = n$ (resp., r is even and $< n$) of the pair (ω, ϕ) corresponds to the rth universal Stiefel-Whitney class.*

(D) We consider briefly a dual situation. For each r $(1 \leq r \leq m)$ let $\bar{V}_{n+m,m}^r = R_{n+m}/R_n \times R_r^m$; each $\bar{V}_{n+m,m}^r \to G_{n+m,n}$ *is an associated bundle of the principal R^m-bundle $\bar{V}_{n+m,m} \to G_{n+m,n}$ with fibre $V_{m,m-r}$.* If we change the order of the base in E_{n+m}, setting $e_i' = e_{n+m-i+1}$, and let R_r' denote the rotation group in the variables $x_1', \cdots, x_r'$, etc., then the natural involutory homeomorphism $\alpha: G_{n+m,n} \to G_{n+m,m}'$ induces the bundle $\bar{V}_{n+m,m}^r \to G_{n+m,n}$ from $V_{n+m,m}'' \to G_{n+m,m}'$.

The Maurer-Cartan forms $\bar{\omega}_{ij} = \omega_{n+m-i+1,n+m-j+1}$ $(1 \leq i, j \leq m)$ are $\bar{V}_{n+m,m}$ basic and define an invariant connection in the R^m-bundle $\bar{V}_{n+m,m} \to G_{n+m,n}$. We define the *dual (r, m)-universal curvature* matrix as the matrix of 2-forms on R_{n+m} whose entries are

$$\bar{\Omega}_{ij}^{(r)} = \sum_{k=r+1}^{n+m} \bar{\omega}_{ik} \vee \bar{\omega}_{kj} \qquad (1 \leq i, j \leq m).$$

The *dual (r, m)-universal Gauss and geodesic curvature forms* $\overline{\Omega}^{(r)}$ and $\overline{\Phi}^{(r)}$ are defined by (4) and (7) of §2, using $\overline{\Omega}_{ij}^{(r)}$ and $\bar{\omega}_{ir}$ (and summations from 1 to m). As in Proposition 2D we find that $\overline{\Omega}^{(r)}$ and $\overline{\Phi}^{(r-1)}$ *are a transgressive pair in the $(r-1)$-sphere bundle* $\overline{V}_{n+m,m}^{r-1} \to \overline{V}_{n+m,m}^{r}$.

Relative to a given subdivision K of $G_{n+m,n}$ we define the chain $\bar{w}_r = (0 \cdots 0\ r)^+ - (0 \cdots 0\ r)^-$ for $1 \leqq r \leqq m$; if $\bar{w}^r$ denotes the corresponding cochain, then $d\bar{w}^r = [1+(-1)^r]\bar{w}^{r+1}$. The *dual R_n-universal Stiefel-Whitney classes* are the cohomology classes $(\bar{w}^r) \in H^r(G_{n+m,n}, Z)$ if r is odd or $r = m$, and $(\bar{w}^r) \in H^r(G_{n+m,n}, Z_2)$ if r is even and $r < m$. $\bar{w}^m$ has infinite order, and $\bar{w}^r$ has order 2 for $r < m$.

We define a *dual admissible pair of sections* $\bar{f}_p: G_{n+m,n} - (m \cdots m\ m-p+1)_* \to \overline{V}_{n+m,m}^p$ for $p = r-1$, r as in Definition 3C; then $(m \cdots m\ m-r)_*$ is the support of $\mathfrak{D}\bar{w}^r$. As dual to Theorem 3C we have the

THEOREM. *Let $\overline{\Omega}^{(r)}$ and $\overline{\Phi}^{(r)}$ denote the dual (r, m)-universal Gauss and geodesic curvature forms; for a dual admissible pair of sections $\bar{f}_p$ let*

$$\bar{\omega} = \bar{f}_r^* \overline{\Omega}^{(r)} \quad and \quad \bar{\phi} = f_{r-1}^* \overline{\Phi}^{(r-1)}.$$

Then $(\bar{\omega}, \bar{\phi})$ is a (Z, r)-pair $(1 \leqq r \leqq m)$ on $G_{n+m,n}$ representing the dual Stiefel-Whitney class $\bar{w}^r$.

4. The tangent classes. (A) Suppose that Y is a smooth orientable Riemann $(n+m)$-manifold and X is an orientable submanifold of dimension n regularly embedded in Y. The connection forms ω_{ij} $(1 \leqq i, j \leqq n+m)$ of Y and its curvature forms Ω_{ij}, forms defined on the bundle $\mathcal{S}_{n+m}(Y)$ of orthonormal $(n+m)$-frames on Y, are entries in skew-symmetric matrices, related by

$$(1) \qquad d\omega_{ij} = \Omega_{ij} + \sum_{k=1}^{n+m} \omega_{ik} \vee \omega_{kj} \qquad (1 \leq 1, j \leq n+m).$$

Consider the subbundle $\mathcal{B}_{n+m}(X)$ of $\mathcal{S}_{n+m}(Y)$ consisting of those frames $(x; e_1 \cdots e_{n+m})$ such that $x \in X$ and $e_1, \cdots, e_n$ are in the tangent space $X(x)$ of X at x. If f denotes the inclusion mapping of these bundles, then the Riemann geometry on X induced from that of Y has connection forms $\theta_{ij} = f^* \omega_{ij}$ and curvature forms

$$(2) \qquad \Theta_{ij} = \Omega_{ij} + \sum_{k=n+1}^{n+m} \omega_{ik} \vee \omega_{kj} \qquad (1 \leqq i, j \leqq n)$$

related by

$$(3) \qquad d\theta_{ij} = \Theta_{ij} + \sum_{k=1}^{n} \theta_{ik} \vee \theta_{kj}.$$

(B) Now let $Y = E_{n+m}$ $(m \geqq n+1)$, and fix an orientation of X. Let $T: X \to G_{n+m,n}$ be the tangent map, defined by letting $T(x)$ be that oriented n-plane in $G_{n+m,n}$ which is parallel to the tangent space $X(x)$ and has the

same orientation. If T^* denotes the induced homomorphism of cohomology rings, there the *tangent Stiefel-Whitney classes $w^r(X)$ of X* are defined as $T^*(w^r)$ for $1 \leq r \leq n$.

REMARK. It is known that $w^1(X) = 0$ if and only if X is orientable. By using forms with twisted coefficients we can prove an analogue of Theorem 4D below, deleting the hypothesis that X is orientable, and obtain an integral representation of $w^1(X)$. That formula expresses the elementary fact that X is orientable if and only if the bundle $\mathcal{S}_n(X)$ is disconnected.

Henceforth we assume that $2 \leq r \leq n$.

The classes $w^r(X)$ can be (in fact, were first) defined without reference to imbedding, in terms of transgression in the bundle $\mathcal{S}_{n-r+1}(X)$ of $(n-r+1)$-frames on X; i.e., as the primary obstruction to constructing a smooth section of $\mathcal{S}_{n-r+1}(X)$ over X. The equivalence of these definitions is proved by using partial sections as in §3C in the commutative diagram

(4)
$$
\begin{array}{ccc}
\mathcal{S}_{n-r+1}(X) & \xrightarrow{T_{r-1}} & V^{r-1}_{n+m,n} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ \ T\ \ } & G_{n+m,n},
\end{array}
$$

where $T_{r-1}(x; e_1 \cdots e_{n-r+1})$ is the element of $V^{r-1}_{n+m,n}$ consisting of $T(x)$ and the image frame of $e_1 \cdots e_{n-r+1}$ in $T(x)$.

(C) DEFINITION. *The rth geodesic and Gauss curvature forms of X are*

(5)
$$
\Phi^{(r)}(X) = T^*_r \Phi^{(r)} \quad and \quad \Omega^{(r)}(X) = T^*_r \Omega^{(r)}.
$$

REMARK. The Riemann structure on X induced from E_{n+m} coincides with that induced by T_0 from the invariant connection (§2C) on $V_{n+m,n}$ using Maurer-Cartan forms. It follows easily that the forms (5) are in effect constructed by substituting θ_{ij} and

$$
\Theta^{(r)}_{ij} = \Theta_{ij} + \sum_{k=r+1}^{n} \theta_{ik} \vee \theta_{kj} \qquad (1 \leq i, j \leq n)
$$

in the expressions (7) and (4) of §2. In particular, *the forms in (5) are intrinsically defined using the Riemann structure of X.* Furthermore, if V is an r-dimensional submanifold of X, then (replacing n by r and $n+m$ by n in (2)) we see that *the forms in (5) induce the usual Gauss and geodesic curvature forms on V*, using the construction in (A).

From the diagram (4) and Proposition 2D we obtain the

PROPOSITION. *If X is an abstract smooth orientable Riemann manifold, then $(\Omega^{(r)}(X), \Phi^{(r-1)}(X))$ is a transgressive pair in the $(r-1)$-sphere bundle $\mathcal{S}_{n-r+1}(X) \to \mathcal{S}_{n-r}(X)$.*

(D) As in §3C we make the

DEFINITION. An *admissible r-pair of frames on X* are smooth sections

$$f_p \colon X - e(f_p) \to \mathcal{S}_{n-p}(X) \qquad \text{for } p = r - 1, r$$

such that $\pi \circ f_{r-1} = f_r$ in their common domain (π is the bundle map $\mathcal{S}_{n-r+1}(X) \to \mathcal{S}_{n-r}(X)$), where $e(f_p)$ is a smooth polyhedron of dimension $\leq n - p - 1$. It has been shown by Stiefel [9, §4] that admissible r-pairs always exist such that $e(f_p)$ are the supports of the characteristic cycles of X defined by f_p.

By an elementary computation of residues analogous to that made in Theorem 3C we have our main result.

THEOREM. *Let X be a smooth orientable Riemann n-manifold and let $\Omega^{(r)}(X)$, $\Phi^{(r)}(X)$ be the rth Gauss and geodesic curvature forms of X. Then for any admissible r-pair of frames on X, $(f_r^* \Omega^{(r)}(X), f_{r-1}^* \Phi^{(r-1)}(X))$ is a $(\mathbf{Z}, r)$-pair on X whose cohomology class is the rth Stiefel-Whitney class of X $(2 \leq r \leq n)$; i.e., for every smooth integral r-chain c such that $|c| \cap e(f_r) = \varnothing$, $|\partial c| \cap e(f_{r-1}) = \varnothing$, we have*

$$(6) \qquad w^r(X) \cdot c = \int_c f_r^* \Omega^{(r)}(X) - \int_{\partial c} f_{r-1}^* \Phi^{(r-1)}(X).$$

Equation (6) is considered a congruence modulo 2 if r is even and $< n$.

REMARK. Suppose again that X is imbedded in E_{n+m}; we ask whether we can use the diagram (4) and Theorem 3C to construct a pair $(T^*\omega, T^*\phi)$ to be used in (6). For an arbitrary position of X in E_{n+m} we see that the singularities of such a pair may not have the right dimensions to satisfy the requirements for a $(\mathbf{Z}, r)$-pair. However, if we imbed X so that T is one-one, then it is easy to see that we can construct partial sections $g_p \colon X \to \mathcal{S}_{n-p}(X)$ so that (1) $T_p \circ g_p = f_p \circ T$ for $p = r-1, r$, and (2) $g_r = \pi \circ g_{r-1}$, where the f_p are an admissible pair of sections as in §3C. Then $g_r^* \Omega^{(r)}(X) = T^*\omega$ and $g_{r-1}^* \Phi^{(r-1)}(X) = T^*\phi$.

(E) Returning to the situation in (A), we can define the normal Stiefel-Whitney classes $\bar{w}^r(X, Y)$ $(1 \leq r \leq m)$ of X in Y (e ˓., as the primary obstructions to constructing smooth sections of appropriate bundles of normal frames to X in Y). By paralleling the development of §3D we obtain the dual of Theorem 4D, *representing $\bar{w}^r(X, Y)$ in terms of normal curvature forms $\bar{\Omega}^{(r)}(X, Y)$ and $\bar{\Phi}^{(r-1)}(X, Y)$.* In case $m = n = r$ and X is compact we obtain Chern's formula [4, p. 683] for Whitney's invariant of X in Y.

If we prescribe a normal frame f on X then the same construction gives an integral representation for the relative Stiefel-Whitney classes $w^r(X, Y, f)$ in terms of curvature forms. These formulas generalize the form of the Gauss-Bonnet Theorem for manifolds Y with boundary X (where f is the inwardly drawn normal vector field); see Allendoerfer-Weil [1].

PROBLEM. In §§2, 3 we have constructed certain partially invariant exterior polynomials in Maurer-Cartan forms of the rotation group, which determine a base for the cohomology ring (mod 2) of a classifying space for R_n;

furthermore, these polynomials could be interpreted as forms in a sequence of bundles associated with the R_n-universal bundle. Can we define a product of these polynomial pairs (presumably in a Whitney sum of associated bundles) and thereby construct an analogue (with mod 2 coefficients) of the theory (with real coefficients) of invariant polynomials of Chern-Weil (see Chern [5, Chapter 3])?

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